29 Calculer l'expression du laplacien en coordonnées polaires.

Soit $f \in \mathcal{C}^2(\mathbb{R}^2)$, $g : (r,\theta) \mapsto f(r\cos\theta, r\sin\theta) \in \mathcal{C}^2(\mathbb{R}^2)$

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ $\qquad f(*) = f(r\cos\theta, r\sin\theta)$.

Soit $(r,\theta) \in \mathbb{R}^2$

$\frac{\partial g}{\partial r}(r,\theta) = \cos\theta \frac{\partial f}{\partial x}(*) + \sin\theta \frac{\partial f}{\partial y}(*)$ puis $\frac{\partial^2 g}{\partial r^2}(r,\theta) = \cos\theta\left[\cos\theta \frac{\partial^2 f}{\partial x^2}(*) + \sin\theta \frac{\partial^2 f}{\partial y \partial x}(*)\right] + \sin\theta\left(\cos\theta \frac{\partial^2 f}{\partial x \partial y}(*) + \sin\theta \frac{\partial^2 f}{\partial y^2}(*)\right)$

$= \cos^2\theta \frac{\partial^2 f}{\partial x^2}(*) + 2\cos\theta\sin\theta \frac{\partial^2 f}{\partial x \partial y}(*) + \sin^2\theta \frac{\partial^2 f}{\partial y^2}(*)$ (Schwarz)

$\frac{\partial g}{\partial \theta}(r,\theta) = -r\sin\theta \frac{\partial f}{\partial x}(*) + r\cos\theta \frac{\partial f}{\partial y}(*)$ puis $\frac{\partial^2 g}{\partial \theta^2}(r,\theta) = -r\cos\theta \frac{\partial f}{\partial x}(*) - r\sin\theta \frac{\partial f}{\partial y}(*)$

$-r\sin\theta\left[-r\sin\theta \frac{\partial^2 f}{\partial x^2}(*) + r\cos\theta \frac{\partial^2 f}{\partial y \partial x}(*)\right]$

$+ r\cos\theta\left[-r\sin\theta \frac{\partial^2 f}{\partial x \partial y}(*) + r\cos\theta \frac{\partial^2 f}{\partial y^2}(*)\right]$

$= -r\cos\theta \frac{\partial f}{\partial x}(*) - r\sin\theta \frac{\partial f}{\partial y}(*)$

$+ r^2\sin^2\theta \frac{\partial^2 f}{\partial x^2}(*) - 2r^2\cos\theta\sin\theta \frac{\partial^2 f}{\partial x \partial y}(*) + r^2\cos^2\theta \frac{\partial^2 f}{\partial y^2}(*)$

Si $r \neq 0$,

$\frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta) = \Delta f(*) - \frac{1}{r}\left[\cos\theta \frac{\partial f}{\partial x}(*) + \sin\theta \frac{\partial f}{\partial y}(*)\right]$

$$\Delta f(*) = \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta)$$