Propriété

$\det A^{\mathsf{T}} = \det A.$

Démonstration

$$\begin{aligned}\det\left(A^{\mathsf{T}}\right) &= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &\underset{j=\sigma(i)}{=} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \prod_{j=1}^{n} a_{\sigma^{-1}(j),j} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) a_{\sigma^{-1}(1),1} \dots a_{\sigma^{-1}(n),n} \\ &\underset{\sigma'=\sigma^{-1}}{=} \sum_{\sigma'\in\mathfrak{S}_n} \varepsilon(\sigma') a_{\sigma'(1),1} \dots a_{\sigma'(n),n} = \det A.\end{aligned}$$

Le dernier changement d'indice étant licite car $\left|\begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & \mathfrak{S}_n \\ \sigma & \longmapsto & \sigma^{-1} \end{array}\right.$ est bijective. $\square$