exercice 18 /

Soit $n \in \mathbb{N}_{\geq 2}$

$$D_n = \begin{vmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_n$$

$$\underset{\%\, C_1}{=} (a+b)\, D_{n-1} - 1 \times \begin{vmatrix} ab & 0 & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & a+b & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_{n-1}$$

$$\underset{\%\, L_1}{=} (a+b)\, D_{n-1} - ab\, D_{n-2}$$

ie $\boxed{\forall n \in \mathbb{N}^*_{\geq 2} \quad D_n = (a+b) D_{n-1} - ab D_{n-2}}$

Suite récurrente d'équation caractéristique $r^2 - (a+b)r + ab = 0$

ie $\underline{(r-a)(r-b) = 0}$

- $\underline{\text{Si } a = b = 0}$: $\forall n \in \mathbb{N}^*$, $\underline{D_n = 0}$ /
- $\underline{\text{Si } a = b \neq 0}$: $\exists \lambda, \mu \in \mathbb{C}$ tq $\forall n \in \mathbb{N}^*$, $\underline{D_n = \lambda a^n + \mu n a^n}$

Or $\begin{cases} D_1 = 2a \\ D_2 = 3a^2 \end{cases} \Leftrightarrow \begin{cases} \lambda a + \mu a = 2a \\ \lambda a^2 + 2\mu a^2 = 3a^2 \end{cases}$

$\Leftrightarrow \begin{cases} \lambda + \mu = 2 \\ \lambda + 2\mu = 3 \end{cases}$

$\Leftrightarrow \begin{cases} \lambda = 1 \\ \mu = 1 \end{cases}$

ie $\forall n \in \mathbb{N}^*$: $\underline{D_n = a^n + n a^n = (n+1) a^n}$ /

- $\underline{\text{Si } a \neq b}$ $\exists \lambda, \mu \in \mathbb{C}$ tq $\forall n \in \mathbb{N}^*$ $D_n = \lambda a^n + \mu b^n$

Or $\begin{cases} D_1 = a+b \\ D_2 = a^2 + b^2 + ab \end{cases} \Leftrightarrow \begin{cases} \lambda a + \mu b = a+b \\ \lambda a^2 + \mu b^2 = a^2 + b^2 + ab \end{cases}$

Remarque pratique : la relation de récurrence incite à poser D_0 tel que $D_2 = (a+b)D_1 - ab D_0$ soit $D_0 = 1$ (logique...)
Ce qui rend tous les calculs plus simples.

$$\iff \begin{cases} \mu = 1 \\ b^2 = b^2 \end{cases} \text{si } a = 0 \quad \text{ou} \quad \begin{cases} \lambda = \dfrac{a+b-\mu b}{a} \text{ si } a \neq 0 \\ \cancel{a^2} \cancel{+ab} - \mu ab + \mu b^2 = \cancel{a^2} + b^2 \cancel{+ab} \end{cases}$$

$$\iff \begin{cases} \lambda = \dfrac{a+b-\mu b}{a} \\ \mu(b^2 - ab) = b^2 \end{cases} \text{si } a \neq 0 \quad \text{ou } \mu = 1 \text{ si } a = 0$$

$$\iff \mu = 1 \text{ si } a = 0 \text{ ou } \lambda = 1 \text{ si } b = 0 \text{ ou } \begin{cases} \lambda = \dfrac{a+b-\frac{b^2}{b-a}}{a} \\ \mu = \dfrac{b^2}{b^2-ab} = \dfrac{b}{b-a} \end{cases} \text{sinon}$$

$$\iff \mu = 1 \text{ si } a = 0 \text{ ou } \lambda = 1 \text{ si } b = 0 \text{ ou } \begin{cases} \lambda = \dfrac{a^2 b}{a^2 b - b^2 a} = \dfrac{a}{a-b} \\ \mu = \dfrac{b}{b-a} \end{cases} \text{sinon}$$

$$\iff \begin{cases} \lambda = \dfrac{a}{a-b} \\ \mu = \dfrac{b}{b-a} = \dfrac{-b}{a-b} \end{cases} \text{car } a \neq b$$

ic $\forall n \in \mathbb{N}^* : D_n = \dfrac{a^{n+1} - b^{n+1}}{a-b}$

Donc : $\boxed{\forall n \in \mathbb{N}^* : D_n = \begin{cases} 0 & \text{si } a = b = 0 \\ (n+1)a^n & \text{si } a = b \neq 0 \\ \dfrac{a^{n+1} - b^{n+1}}{a-b} & \text{sinon} \end{cases}}$