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1) $S_n = \sum_{k=0}^{n} \frac{k}{n^2} \cos \frac{k\pi}{n}$

$= \frac{1}{n} \sum_{k=0}^{n} \frac{k}{n} \cos\left(\frac{k}{n}\pi\right)$

$= \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) - \frac{1}{n}$

avec $f : x \longmapsto x\cos(x\pi) \in \mathcal{C}([0;1])$

$S_n \longrightarrow \int_0^1 \overbrace{x}^{I'} \underbrace{\cos(x\pi)}_{I} \, dx$

$= \left[x \frac{\sin(x\pi)}{\pi}\right]_0^1 - \frac{1}{\pi}\int_0^1 \sin(x\pi)\,dx$

$= + \frac{1}{\pi}\left[+ \frac{\cos(x\pi)}{\pi}\right]_0^1$

$= \frac{1}{\pi^2}(-1-1)$

$= \frac{-2}{\pi^2}.$

$\boxed{\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f \quad \text{si } f \in \mathcal{C}_m([0,1])}$

$\boxed{+ \text{ 22 pour demain}}$

2) $\sum_{k=0}^{n} \frac{n}{n^2+k^2} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1+\left(\frac{k}{n}\right)^2} + \frac{1}{2n}$

$\longrightarrow \int_0^1 \frac{dx}{1+x^2} = \left(\text{Arctan } x\right)_0^1 = \frac{\pi}{4}$

car $x \longmapsto \frac{1}{1+x^2} \in \mathcal{C}([0,1])$

3) $\sum_{k=0}^{n} \frac{k}{n^2+k^2} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2} + \frac{1}{2n}$

$\longrightarrow \int_0^1 \frac{x}{1+x^2}dx = \left[\frac{\ln(1+x^2)}{2}\right]_0^1$

$= \ln 2 / 2$

4) $\displaystyle\sum_{k=0}^{n} \frac{1}{\sqrt{n^2+2kn}} = \frac{1}{n}\sum_{k=0}^{n} \frac{1}{\sqrt{1+2\frac{k}{n}}}$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sqrt{1+2\frac{k}{n}}} + \boxed{\frac{1}{n\sqrt{3}}}$$

$$\longrightarrow \int_0^1 \frac{2\,dx}{2\sqrt{1+2x}} \qquad \text{“}\frac{u'}{2\sqrt{u}}\text{”}$$

Car $x \mapsto \dfrac{1}{\sqrt{1+2x}} \in \mathcal{C}([0;1])$

dx

$$\sum_{k=0}^{n} \frac{1}{\sqrt{n^2+2kn}} \longrightarrow \left[\sqrt{1+2x}\right]_0^1 = \sqrt{3}-1.$$

5) $\displaystyle\sum_{k=0}^{n} \frac{k^2}{n^3+nk^2} = \frac{1}{n}\sum_{k=0}^{n} \frac{(k/n)^2}{1+(k/n)^2}$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \frac{(k/n)^2}{1+(k/n)^2} + \frac{1}{2n}$$

$$\longrightarrow \int_0^1 \frac{x^2+1-1}{1+x^2}\,dx = 1-\left[\operatorname{Arctan} x\right]_0^1$$

$$= 1-\pi/4$$

$$= \frac{4-\pi}{4} > 0.$$

Car $x \mapsto \dfrac{x^2}{1+x^2} \in \mathcal{C}([0;1])$

6) $\displaystyle \ln\left[\left(\frac{(2n)!}{n^n\, n!}\right)^{\frac{1}{n}}\right] = \frac{1}{n}\ln\frac{(n+1)(n+2)\ldots(2n)}{n\times n\times\ldots\times n}$

$$= \frac{1}{n}\ln\left(\prod_{k=1}^{n}\frac{n+k}{n}\right) = \frac{1}{n}\sum_{k=\cancel{1}\,0}^{n}\ln\left(1+\frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\ln\left(1+\frac{k}{n}\right) + \frac{\ln 2}{n} \longrightarrow \int_0^1 \ln(1+x)\,dx$$

Car $x \longmapsto \ln(1+x) \in \mathcal{C}([0,1])$

$$\int_0^1 1 \times \ln(1+x)\,dx = \left[x\ln(1+x)\right]_0^1 - \int_0^1 \frac{x^{+1-1}}{1+x}\,dx$$

$x \mapsto x$

$x \mapsto \ln(1+x)$

$\mathcal{C}^1([0;1])$

$$= \ln 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right)dx$$

$$= \ln 2 - \left[x - \ln(1+x)\right]_0^1$$

$$= 2\ln 2 - 1$$

$$\boxed{\left(\frac{(2n)!}{n^n\, n!}\right)^{\frac{1}{n}} \longrightarrow e^{2\ln 2 - 1} = \frac{4}{e}.}$$

car exp $\mathcal{C}^0$ en $2\ln 2 - 1$.